EMP support in JuMP

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Outline

1. What is Extended Mathematical Programming (EMP)
2. EMP solver
3. Modeling of EMP problems in JuMP

What is EMP?

EMP (Extended Mathematical Programming) is a framework to model optimization problem that do not fit into "classical" optimization problem via simple annotations

EMP in GAMS

- Multi-players problems (MOPEC): (Generalized) Nash Equilibrium
- Complementarity Problems and Variational Inequalities
- Bilevel Problems (MPEC)
- Multistage stochastic programming, Disjunctive Programs

Motivation for EMP support

Motivations & Goals for designing a tool

- Ease of use for non-experts: the modeler uses simple annotations
- The model is augmented/transformed programmatically
- Provide different schemes for solving the problem
- Pass/Exploit the structure of the problem (feasible set)
- No errors/typos during the reformulation
- Allow for easy comparisons between schemes

JAMSDWriter.jl: MPB Solver

- Loosely based on `AmplNLWriter.jl`
- Standard LQP/NLP solver
- Translate the EMP data structure

EMP.jl: modeling package

- Equilibrium support
- Optimal Value Function (OVF) support

VI background

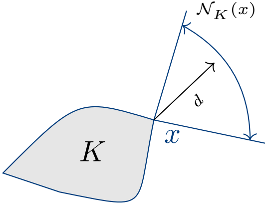
VI of the first kind

$F\colon \mathbb{R}^n \to \mathbb{R}^n$, Z convex subset of $\mathbb{R}^n$. Find $z \in Z$ such that

$$\langle F(z), z - y \rangle \geq 0 \qquad \forall y \in Z$$

If $F = \nabla f$ or $F = (\nabla_x L, -\nabla_\lambda L)$, VI stems from an optimization problem

Normal cone operator: $N_K(x) := \{d \mid \langle y - x, d \rangle \leq 0 \quad \forall y \in K\}$.



$$d \in \mathcal{N}_K(x) \qquad VI(F, K) \equiv 0 \in F(z) + N_K(z)$$

Complementary Problem $0 \leq x \perp F(x) \geq 0 \quad \equiv \quad 0 \in F(x) + N_{\mathbb{R}^n_+}(x)$

where $x \perp F(x) \quad \equiv \quad x^T F(x) = 0$

(Generalized) Nash Equilibrium Problem [(G)NEP]

For each agent i: x_i is the decision variable, x_{-i} is the other variables

$$\begin{aligned} \underset{x_i}{\text{minimize}} \quad & f_i(x_i, x_{-i}) \\ \text{subject to} \quad & g_i(x_i, x_{-i}) \in K_i \end{aligned}$$

K_i convex cone ($\mathbb{R}^n_+$, $\mathbb{R}^n_-$, $\{0\}$, SOC, ...)

example of coupling constraints

- shared resources among agents: $\sum_{i=1}^N x_i \leq u$
- strategic decisions

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- strategic decisions

- No hierarchy between players, (otherwise MPECs or EPECs)
- An example of a VI agent: market clearing conditions

$$0 \leq \text{supply} - \text{demand} \quad \perp \quad \text{price} \geq 0$$

MOPEC is solved by simultaneously solving all the KKT systems

First Order Optimality Conditions under CQ

$$\begin{array}{ll} \underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}) \\ \text{subject to} & g_i(x_i, x_{-i}) \in K_i \end{array} \Longrightarrow \begin{array}{c} \nabla_{x_i} f_i(x_i, x_{-i}) + \langle \nabla_{x_i} g_i(x_i, x_{-i}), \lambda \rangle = 0 \\ K_i^* \ni \lambda_i \perp g_i(x_i, x_{-i}) \in K_i \end{array}$$

Solution methodology for GNEP

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- Resulting MCP: $X \ni z \perp F(z) \in X^*$
- X is a simply bounded set
- F is non-monotone in general
- PATH or PATHAVI can solve many NEP instances (GNEP more challenging)

Problem type

Objective function		or	Constraint	
$\underset{x}{\text{minimize}}$	$f(x) + \boldsymbol{\rho(F(x))}$		$\underset{x}{\text{minimize}}$	$f(x)$
			subject to	$g(x) \leq 0$
subject to	$g(x) \leq 0$			$\boldsymbol{\rho(F(x)) \leq \alpha}$

$$\rho(v) := \sup_{u \in U} \langle G(v), u \rangle - k(u)$$

$G(v) := Bv + b$, k is convex, U polyhedral

Aravkin et al. [2013]

Quadratic Support (QS) examples

Fitting Examples: apply ρ to $F(x)$ — Aravkin et al. [2013]

	ℓ_1	ℓ_2	Huber	Hinge Loss
U	$[-1,1]^m$	0	$[-\lambda,\lambda]^m$	$[0,1]^m$
Q	0	$\frac{1}{2}I$	$\frac{1}{2}I$	0
$G(y)$	y	y	y	$y-\varepsilon$

and more: soft Hinge, Vapnik, Hubnik, elastic net

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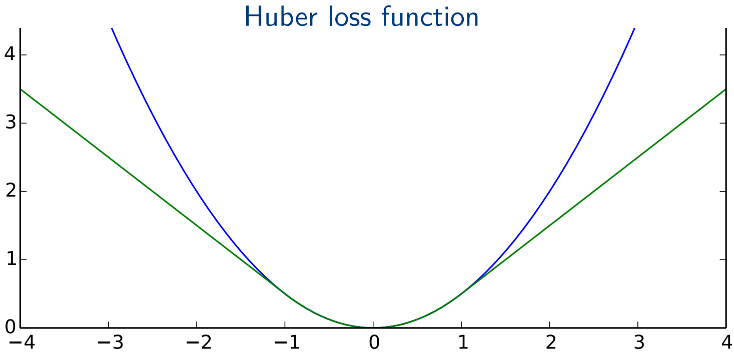
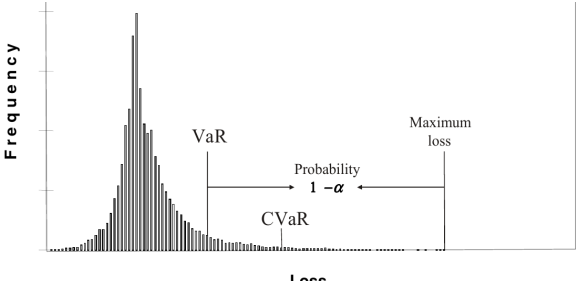


Coherent Risk Measure Examples — Shapiro et al. [2009]

- Expected value: $\mathbb{E}[Z] = \sigma_{\{p_k\}}(Z)$
- Upper CVaR (AV@R, expected shortfall): $\mathrm{CVaR}_{1-\alpha}(z) = \sigma_{D(\alpha)}(Z)$
  with $\mathcal{D}(\alpha) := \left\{ y \in \prod_k [0, \alpha^{-1}] \mid \mathbb{E}[y] = 1 \right\}$
- Algebra of support function: $\lambda\mathbb{E}[z] + (1-\lambda)\mathrm{CVaR}_{1-\alpha}(z)$ is the support function of $\lambda\{p_k\} + (1-\lambda)\mathcal{D}(\alpha)$



Saddle-point reformulation

Game with payoff function: $\bar{L}(x,y) = f(x) + y^T G(F(x)) - k(y)$

$$\min_{x \in X} \bar{L}(x,y) \qquad \max_{y \in \bar{Y}} \bar{L}(x,y)$$

Dual problem (QP)

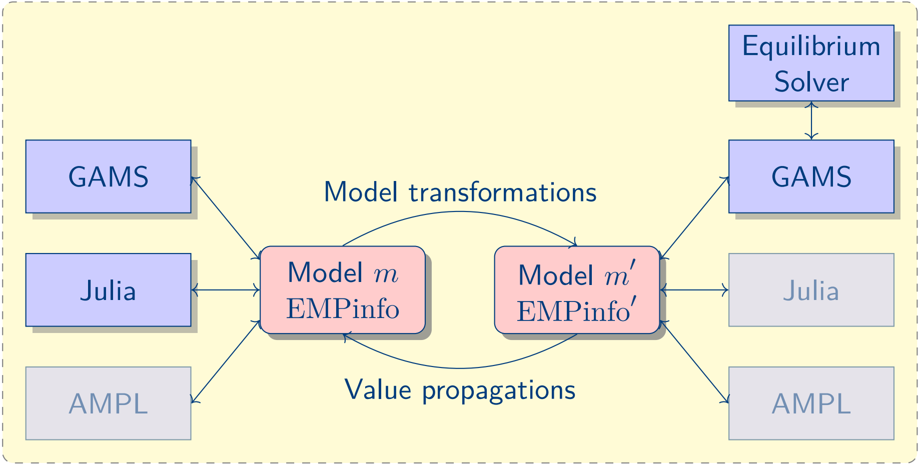
Find a minimization problem with the same objective function value

Conjugate reformulation

- ρ (Fenchel) conjugate of $k + \delta_U$: $\rho(u) = \inf_{u=u_1+u_2} k^*(u_1) + \sigma_U(u_2)$
- $k(u) = u^T Q u = \|L^T u\|_2$ (Q psd)

$$\rho(F(x)) = \inf_s \ \tfrac{1}{2}\|s\|_2^2 + \sigma_U(F(x) - Ls)$$

EMP solver



The model representation inside the EMP solver is independent of any model language

EMP info storage: mathematical programm

Optimization Problem definition

- Mathematical Programm (MP) is the basic building block
- The optimization problem is described by a graph
- Graph with 2 types of nodes: MP or Equilibrium (collection of MPs)
- Examples: bilevel, MPEC, EPEC, MOPEC

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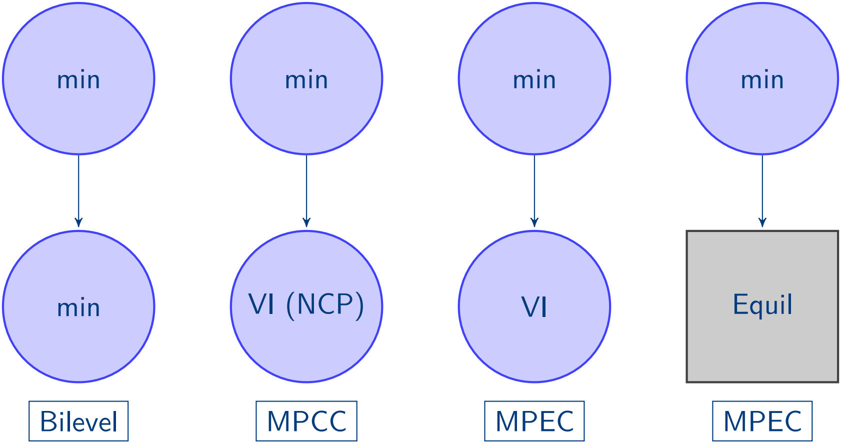
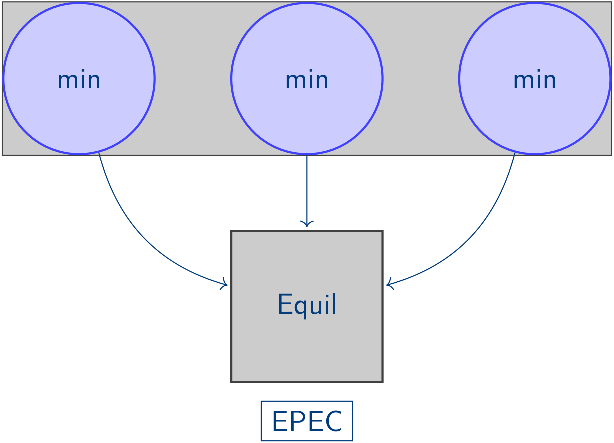
- Easy extraction of subproblems (splitting methods (diag, Gauss-Seidel))
- Fix variable values, simply expressions.



min
min
min
min
min
VI (NCP)
VI
Equil
Bilevel
MPCC
MPEC
MPEC

Example II



Model object

- JuMP container for variables and equations
- list of Mathematic Programms
- list of Equilibriums
- list of equations (for global indices)
- list of special variables

Mathematical Programm (MP)

- variables and equations indices
- complementary information $(0 \leq F(x) \perp x \geq 0)$
- objective variable / objective equation / sense
- MP in MP
- equilibriums in MP

EMP.jl API example I


```
mopec = EMP.Model()

ag = MathPrgm(mopec)
mkt = MathPrgm(mopec)

EquilibriumProblem(mopec, [ag, mkt])

n = 3

@variableMP(mkt, y >= 0)
@variableMP(ag, x[1:n] >= 0)
@variableMP(mkt, p[1:n] >= 0)

JuMP.fix(p[2], 1.)
JuMP.setvalue(x[1:n], ones(n))

vipair(mkt, [b[i] + ATmat[i]*y - x[i] for i=1:n], p)
vipair(mkt, sum(-ATmat[i]*p[i] for i=1:n), y)

@constraintMP(ag, sum(p[i]*x[i] for i=1:n) <= sum(p[i]*b[i] for i=1:n) )
@NLobjectiveMP(ag, :Max, sum(s[i] * log(x[i]) for i=1:n))

solveEMP(mopec)
```

Monolitic approach

- A unique container for variables and equations
- the variables and equations are assigned to a MP

⊕ Easy to defined shared variables or equations

⊖ challenges in modeling: what is a natural way to create the partition?

MP-based approach

- Each MP has its own JuMP Model
- The collection of MP in the graph defines the optimition problem

⊕ ownership is trivial

⊖ shared variables/equations are difficult to define

⊖ consistency of the whole model needs to be checked

⊖ Global indices for variables and equations

Future work

EMP solver

- Add automatic/symbolic differentiation support
- Add ability to use Julia to solve the optimization problem https://github.com/mlubin/cmpb

EMP.jl

- User-defined OVF functions
- Define UI/API for hierarchical optimization problems

A. Y. Aravkin, J. V. Burke, and G. Pillonetto. Sparse/robust estimation and kalman smoothing with nonsmooth log-concave densities: modeling, computation, and theory. *Journal of Machine Learning Research*, 14(1): 2689–2728, 2013.

R. T. Rockafellar. Lagrange multipliers and optimality. *SIAM Review*, 35(2): 183–238, 1993.

R. T. Rockafellar. Extended nonlinear programming. In G. Di Pillo and F. Giannessi, editors, *Nonlinear Optimization and Applications 2*, pages 381–399. Springer, 2000.

A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on Stochastic Programming. Modeling and Theory*. SIAM Publishing, Philadelphia, Pennsylvania, 2009.

AFW '16 Electricity Generation example

- Electricity generation example with $\mathrm{CVaR}\left(\cdot\right)$ in the objective function
- Equilibrium problem with 3 agents and 10 scenarios
- VI (MCP) and conjugate better than dual QP (primal)