PiecewiseLinearOpt.jl

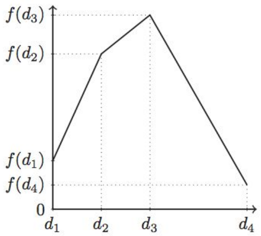
Solving optimization problems containing piecewise linear functions

Linear optimization (LP)

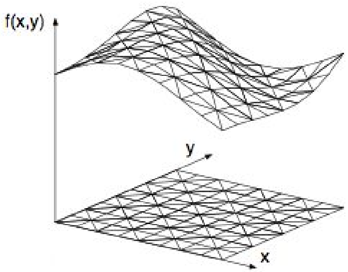
$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

- Surprisingly powerful modeling capabilities
- Can solve large instances, quickly

Piecewise linear (PWL) optimization



$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & Ax \le b \end{aligned}$$

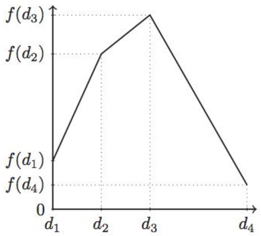


Where

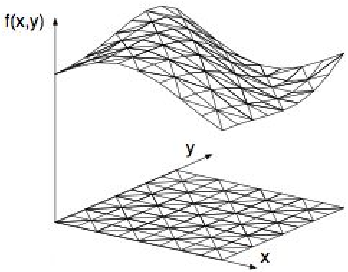
$$f(x) = \begin{cases} c_1^T x + b_1 & x \in P^1 \\ c_2^T x + b_2 & x \in P^2 \\ \vdots & \\ c_d^T x + b_d & x \in P^d \end{cases}$$

- Appears in economics, operations, engineering, …
- Also, “simple” LP approximation for nonlinear optimization

Piecewise linear (PWL) optimization



$$\min_{x} \quad f(x)$$
$$\text{s.t.} \quad Ax \leq b$$

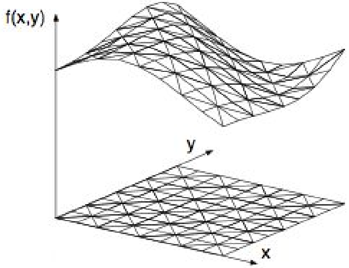


Where

$$f(x) = \begin{cases} c_1^T x + b_1 & x \in P^1 \\ c_2^T x + b_2 & x \in P^2 \\ \vdots & \\ c_d^T x + b_d & x \in P^d \end{cases}$$

- When f is convex, exists canonical transformation to LP
- What about when f is nonconvex?

Mixed-integer optimization (MIP)



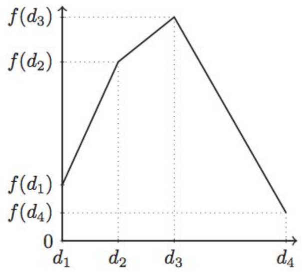
- Discrete variables let you to model discrete decisions
- Discrete decision for PWL function: which piece you are on

$$(x, y) \in \left\{(x, c_1^T x + b_1) : x \in P^1\right\} \cup \left\{(x, c_2^T x + b_2) : x \in P^2\right\} \cup \cdots \cup \left\{(x, c_d^T x + b_d) : x \in P^d\right\}$$

- Standard techniques for modeling logic with MIP ➔ d auxiliary binary variables
  - Almost never what you want!
- 10+ MIP formulations in literature for univariate functions (!)
- Even more complex for higher dimensions

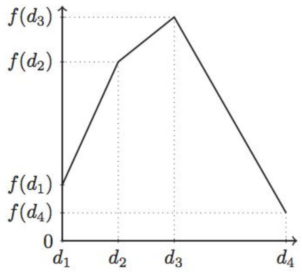
What MIP formulations are there?

Logarithmic formulation (J.P. formulation)



- For univariate PWL functions $f : \mathbb{R} \to \mathbb{R}$
- One particularly small, strong formulation
- Built around Gray codes: sequence of distinct vectors $\{v^i\}_{i=1}^{N-1} \subseteq \{0,1\}^{\lceil \log_2(N-1) \rceil}$ with $|v^i - v^{i+1}| = 1$ and $v^0 = v^1, \ v^N = v^{N-1}$
- Take $L^j = \left\{ i \in [N] : v_j^{i-1} + v_j^i \geq 1 \right\}$, $R^j = \left\{ i \in [N],: v_j^{i-1} + v_j^i \leq 1 \right\}$ for each $j \in [\lceil \log_2(N-1) \rceil]$
- Take univariate PWL function given by breakpoints $\{(x^i, y^i)\}_{i=1}^N$

Logarithmic formulation (J.P. formulation)



$$x = \sum_{i=1}^{N} x^i \lambda_i$$

$$y = \sum_{i=1}^{N} y^i \lambda_i$$

$$\sum_{i \in [N] \setminus L^j} \lambda_i \leq z_j \qquad \forall i \in [\lceil \log_2(N-1) \rceil]$$

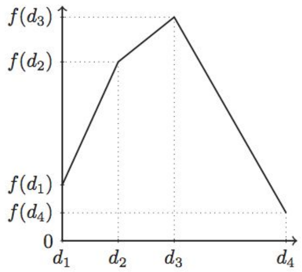
$$\sum_{i \in [N] \setminus R^j} \lambda_i \leq 1 - z_j \quad \forall i \in [\lceil \log_2(N-1) \rceil]$$

$$\sum_{i=1}^{N} \lambda_i = 1$$

$$\lambda \geq 0$$

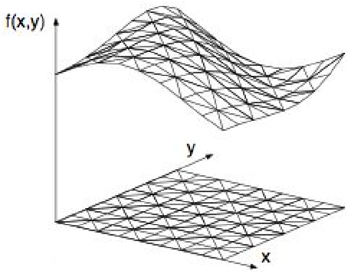
$$z \in \{0,1\}^{\lceil \log_2(N-1) \rceil} y \in H$$

Logarithmic formulation (J.P. formulation)



- What we need:
  1. Understand the paper
  2. Construct a Gray code
  3. Construct the sets $\{(A^j, B^j)\}_{j=1}^{\lceil \log_2(N-1) \rceil}$
  4. Find the breakpoints (if you have a functional form)
- Not impossible, but non-trivial
- Anecdotally, a real barrier for practitioners
- And this is just univariate functions!

2-dimensional PWL functions



- How do we partition the domain?
- Apply gridding along each dimension, triangulate each subrectangle
- Much richer combinatorial structure
- Even harder to model with MIP
- Some formulations work for “regular” triangulations only

`PiecewiseLinearOpt.jl`

1. Representations for arbitrary, multidimensional piecewise linear functions
    - Helper constructors to go from functional form ➔ “breakpoint” representation
2. JuMP extension to build MIP models for 1D and 2D PWL functions
    - 10 formulations for 1D
    - 9 formulations for 2D

PiecewiseLinearOpt.jl

```
julia> UnivariatePWLFunction(1:5, [1,2,4,7,11])
# => PWLFunction{1}([(1.0,),(2.0,),(3.0,),(4.0,),(5.0,)],
                   [1.0,2.0,4.0,7.0,11.0],
                   [[1,2],[2,3],[3,4],[4,5]],
                   Dict())
```

PiecewiseLinearOpt.jl

```
julia> UnivariatePWLFunction(1:5, sin)
# => PWLFunction{1}([(1.0,),(2.0,),(3.0,),(4.0,),(5.0,)],
                   [0.8414,0.9092,0.141,-0.7568,-0.9589],
                   [[1,2],[2,3],[3,4],[4,5]],
                   Dict())
```

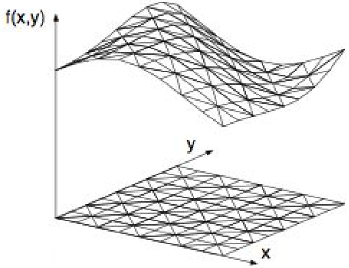
PiecewiseLinearOpt.jl

```
julia> BivariatePWLFunction(1:5, 1:5, (x,y)->(x-1/3)^2+3(y-4/7)^4)
# => PWLFunction{2}([(1.0,1.0),(2.0,1.0),…,(4.0,5.0),(5.0,5.0)],
                    [0.545652,2.87899, …,1167.36,1175.7],
                    [[1,6,2],[2,6,7],…,[19,24,20],[20,24,25]],
                    Dict(:structure=>:BestFit))
```

PiecewiseLinearOpt.jl

```
julia> BivariatePWLFunction(1:5, 1:5, (x,y)->(x-1/3)^2+3(y-4/7)^4)
# => PWLFunction{2}([(1.0,1.0),(2.0,1.0),…,(4.0,5.0),(5.0,5.0)],
                    [0.545652,2.87899, …,1167.36,1175.7],
                    [[1,6,2],[2,6,7],…,[19,24,20],[20,24,25]],
                    Dict(:structure=>:BestFit))
```

MIP formulations for grid triangulations



- State-of-the-art, circa 2014:
  - “Universal” MIP formulation approaches (# of binaries = # of triangles)
  - Small formulations for highly structured triangulations (# of binaries = log(# of triangles)+1)
- In 2015:
  - J.P. shows a small formulation for looser structure (# of binaries = log(# of triangles)+2)
- In 2016:
  - Small formulation for any triangulation (# of binaries = log(# of triangles)+9)
- In 2017:
  - Whittled down to log(# of triangles)+6
- Constants have appreciable affect on performance

Piecewise linear functions in JuMP

```
using JuMP, PiecewiseLinearOpt, Gurobi
model = Model(solver=GurobiSolver())
@variable(model, x)
d = 0:(pi/4):2pi
z = piecewiselinear(model, x, d, sin)
# = piecewiselinear(model, x, UnivariatePWLFunction(d,sin))
@objective(model, Max, z)
```

Grid triangulations in JuMP

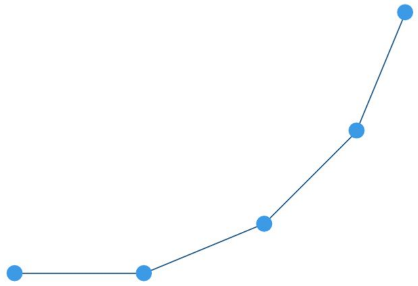
```
using JuMP, PiecewiseLinearOpt, Gurobi
model = Model(solver=GurobiSolver())
@variable(model, x)
d = 0:(pi/4):2pi
z = piecewiselinear(model, x, d, sin, method=:Incremental)
# = piecewiselinear(model, x, UnivariatePWLFunction(d,sin))
@objective(model, Max, z)
```

New research directions (i.e. open PRs)

“Optimal” MIP formulations

- The “constant” term for grid triangulations can make a difference of up to 3-5x
- Can find the “optimal” formulation (smallest # of binaries) by solving a MIP!
- MIP is not very scaleable (yet), but…
- Optimal MIP formulations is best performer, once computed
- Somewhat surprising: loses grid structure along each dimension (x and y)
- Trivial to implement in JuMP:
    - Solve a MIP in JuMP during model generation

Moment curve formulations



- Relaxed notion of “MIP formulation”
  a. Combines:
     i. Algebraic relaxation for (convex hull of) disjunctive set, with auxiliary integer variables
     ii. Constraint programming -like treatment of control variables ($y \in H$)
- E.g. embed disjunctive sets along moment curve
  a. Have inequality description for algebraic description
  b. Treat control variables with custom branching scheme
- Implement in JuMP (and CPLEX) using:
  a. Branching callbacks
  b. Incumbent callbacks

Future directions

- JuMP models for...
  - Higher dimensional PWL functions
  - More complex partitions of domains (not just triangulations)
- Connections with similar problems
  - Convex envelopes for bilinear (and multilinear) functions
  - Approximations for structured nonlinear functions
    - Signomial programming
    - Difference of convex programming