Topology Optimization and JuMP

Immense Potential and Challenges

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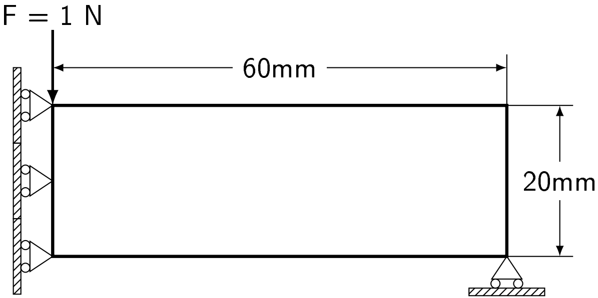
Introduction

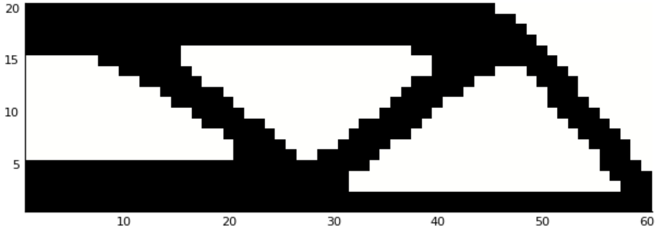
About Me

- First year PhD student at UNSW Canberra
  - Multidisciplinary design optimization lab
  - Supervisor: Tapabrata Ray
- Background:
  - BSc mechanical engineering
  - MSc industrial engineering
- Research interests:
  1. Topology optimization **algorithms**
  2. Topology optimization and finite element **modelling**
  3. Multigrid methods and **scalable** topology optimization
- NumFOCUS GSoC student
  - Locally optimal block preconditioned conjugate gradient (LOBPCG) in IterativeSolvers.jl
  - Buckling analysis

Structure Design



Structure Design



Problems

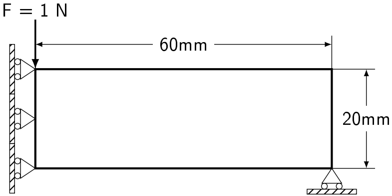
Problem Families

Variables:

1. Each optional mesh element corresponds to a variable,
2. Can be binary or continuous (variable thickness sheet)

Objectives:

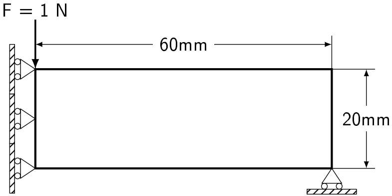
1. Compliance minimization,
2. Material volume/cost minimization,
3. Maximum stress minimization, or
4. Minimum eigenvalue maximization.



Problem Families

Constraints:

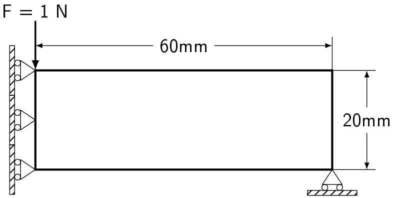
1. Volume constraint,
2. Maximum compliance constraint,
3. Maximum displacement constraint,
4. Local/global stress constraints,
5. Fatigue constraints,
6. Global stability constraints, and/or
7. Others.



Problem Families

Mechanical systems:

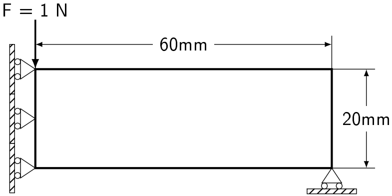
1. Linear, elastic, quasi-static system,
2. Nonlinear, compliant mechanism,
3. Nonlinear, elasto-plastic system,
4. Linear/nonlinear vibrating system, or
5. Others.



Problem Families

Loads:

1. Single or multiple,
2. Static or dynamic,
3. Deterministic or stochastic, and
4. Design-dependent or design-independent.



Pipelines

Topology Optimization Pipelines

- User's Pipeline
  1. Problem context definition
     - Initial design in mesh form
     - Boundary conditions
     - Fixed cells, not allowed to change
     - Defined programmatically or through .inp and similar files
  2. Objective and constraint selection
  3. Topology optimization algorithm
     - Nested Analysis and Design (NAND), or
     - Simultaneous Analysis and Design (SAND)

Topology Optimization Pipelines

- Nested Analysis and Design (NAND) Pipeline
  1. Decide material distribution variables
     - Does this element exist or not?
     - Binary $\in \{0, 1\}$, or relaxed $\in [0, 1]$
  2. FEA
     - Cannot fully remove an element (numerical instability)
     - $x_{soft} = x(1 - x_{min}) + x_{min}$, $x_{min} = 0.001$
     - Makes use of matrix-free linear system and eigenvalue solvers
     - Can be GPU-accelerated or distributed on many computers
  3. Objective and constraint values and derivatives
     - Adjoint method: differentiating through the analysis equations
     - When binary constraints are relaxed, "penalized" variables, i.e. $x_{penal} = x_{soft}^p$ are often used, for some known penalty typically $1 < p \leq 5$.
  4. Update material distribution **and repeat**
     - Optimization magic!

Topology Optimization Pipelines

- Simultaneous Analysis and Design (SAND) Pipeline
  1. FEA
     - Formulate the analysis equations as constraints
     - Analysis variables are decision variables
  2. Optimization modelling
     - Write the composite analysis-design problem as a bigger optimization problem
     - Material distribution decision variables $\boldsymbol{x}$
     - Analysis decision variables, e.g. nodal displacements $\boldsymbol{u}$
     - Analysis constraints, e.g. $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$ and $\boldsymbol{K} = \sum_e \boldsymbol{x}_{penal,e} \boldsymbol{K}_e$
     - Design constraints, e.g. $\sigma_e^v \leq \sigma_y, \forall e$
  3. Optimization magic!
     - Single pass

Examples

Example Problem 1

- Volume constrained compliance minimization
  - Analysis: 1b and 1c
  - Design: 1a and 1d
  - Cheq filter not shown

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad C = \boldsymbol{u}^T \boldsymbol{K} \boldsymbol{u} \tag{1a}$$

subject to

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}, \tag{1b}$$

$$\boldsymbol{K} = \sum_e \rho_e^p \boldsymbol{K}_e, \tag{1c}$$

$$\sum_e v_e x_e \leq V, \tag{1d}$$

$$x_e \in \{0, 1\} \quad \forall e \tag{1e}$$

C: Compliance, convex in $\boldsymbol{x}$

$\boldsymbol{u}$: Displacements

$\boldsymbol{K}$: Global stiffness matrix

$\boldsymbol{f}$: Load vector

$\boldsymbol{K}_e$: Element stiffness matrix e

v_e: Volume of element e

x_e: Does element e exist?

ρ_e: Soft $x_e := x_e(1 - x_{min}) + x_{min}$

V: Volume threshold

p: Known as the "penalty" typically $\in [1, 5]$

Example Problem 2

- Stress constrained volume minimization
  - σ_{ij}^e is the stress tensor inside element e **linear** in $\boldsymbol{u}$
  - $(\sigma_e^v)^2 x_e \leq \sigma_y^2 x_e$ is also a valid constraint since $\sigma_e^v \geq 0$ and x_e is binary
  - $(\sigma_e^v)^2 x_e \leq \sigma_y^2 x_e$ is **bi-convex** in $\boldsymbol{u}$ and x_e

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \sum_e v_e x_e \\ \text{subject to} \quad & \\ & (1b), \\ & (1c), \\ & \sigma_e^v x_e \leq \sigma_y x_e \quad \forall e, \\ & x_e \in \{0, 1\} \quad \forall e \end{aligned}$$

$$\sigma_e^v := \left(\frac{1}{2}(\sigma_{11}^e - \sigma_{22}^e)^2 + \frac{1}{2}(\sigma_{22}^e - \sigma_{33}^e)^2 + \frac{1}{2}(\sigma_{33}^e - \sigma_{11}^e)^2 + 3(\sigma_{12}^e)^2 + 3(\sigma_{23}^e)^2 + 3(\sigma_{31}^e)^2 \right)^{\frac{1}{2}} \quad \forall e$$

σ_y: yield stress of the material

Example Problem 3

- Buckling constrained volume minimization
  - Positive semidefinite constraint
  - $\boldsymbol{K}$ is a **linear** function of $\boldsymbol{x}$
  - $\boldsymbol{K}_\sigma$ is a **bi-linear** function of $\boldsymbol{u}$ and $\boldsymbol{x}$

$$\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & \sum_e v_e x_e \\
\text{subject to} \quad & \\
& (1b), \\
& (1c), \\
& \boldsymbol{K}_\sigma = \sum_e x_e \int_{\Omega_e} \boldsymbol{G}^{e^T} \psi^e \boldsymbol{G}^e dV, \\
& \boldsymbol{K} + \lambda_s \boldsymbol{K}_\sigma \succcurlyeq 0, \\
& x_e \in \{0, 1\} \quad \forall e
\end{aligned}$$

$\boldsymbol{K}_\sigma$: Stress stiffness matrix

λ_s: Load multiplier under which design must be stable

$\boldsymbol{\sigma}^e$: matrix form of σ_{ij}^e from the previous slide

$\psi^e := \text{kron}(\boldsymbol{I}_{dim \times dim}, \boldsymbol{\sigma}^e)$

$\boldsymbol{G}^e$: basis function derivatives of element e arranged in a special order

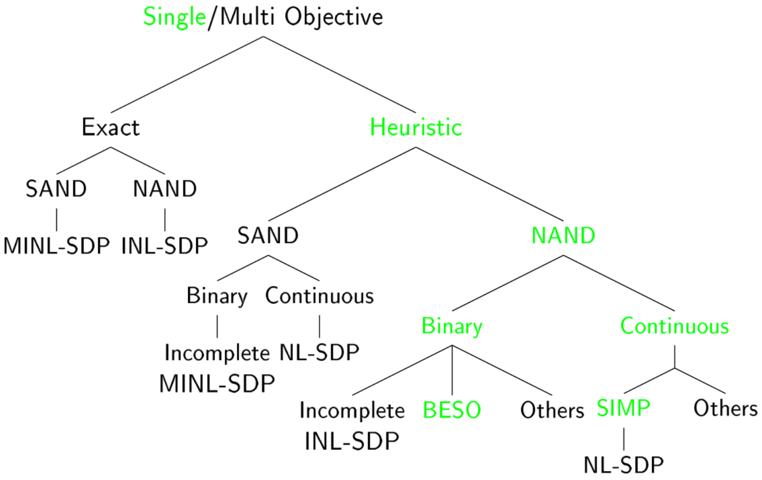
Algorithms

Algorithm Classification Tree

Acronyms:

- **SAND**: Simultaneous analysis and design
- **NAND**: Nested analysis and design
- **MINL-SDP**: Mixed integer nonlinear and semidefinite programming
- **INL-SDP**: Integer nonlinear and semidefinite programming (no continuous variables)
- **NL-SDP**: Nonlinear and semidefinite programming
- **SIMP**: Solid isotropic material with penalization
- **BESO**: Bi-directional evolutionary structural optimization [1]

Algorithm Classification Tree



Why am I here?

Next Generation Topology Optimization

- Continuous and binary variables
- Flexible constraint handling
  - Block constraints with Jacobian of fixed sparsity pattern
  - Bi-linear, bi-convex and nonlinear constraints
  - Conic constraints
  - Partial structure, e.g. some bi-linear and some bi-convex
- Linear time and memory complexity
  - Can have 100s of millions of variables
- Numerically robust to scaling
- Scalable optimization pipeline (pre-processor and solver)
  - Efficiently GPU-accelerated
  - Efficiently distributed to multiple machines
- Single- and multi- objective

Next Generation Topology Optimization

Possible with Julia's optimization ecosystem?

Next Generation Topology Optimization

What can I offer?

- Prayers!
- Oh and I am ready to code (after paper submissions and GSoC!).

Demo

Demo

Further Readings I

[1] Xiaodong Huang and Yi Min Xie. A further review of ESO type methods for topology optimization. *Structural and Multidisciplinary Optimization*, 41(5):671–683, 2010.

[2] Xia Liu, Wei-Jian Yi, Q.S. Li, and Pu-Sheng Shen. Genetic evolutionary structural optimization. *Journal of Constructional Steel Research*, 64(3):305–311, 2008.

[3] K Svanberg. The method of moving asymptotes - a new method for structural optimization. *International Journal for Numerical Methods in Engineering*, 24(2):359–373, 1987.

[4] Krister Svanberg. A Class of Globally Convergent Optimization Methods Based on Conservative Convex Separable Approximations. *SIAM Journal on Optimization*, 12(2):555–573, 2002.

Questions?

Extras

Method of moving asymptotes (MMA)

- Most popular nonlinear programming algorithm used in topology optimization
- Sequential convex programming
- First order approximation of f with respect to $\frac{1}{x-L}$ or $\frac{1}{U-x}$, whichever is convex given the sign of $f'(x)$
- Originally proposed in [3]
- Later improved and similar algorithms were proposed in [4]
- Dual algorithm is fully separable so it can be GPU-accelerated and distributed
- Only handles inequality constraints

Adjoint method

- $\rho_e = x_e(1 - x_{min}) + x_{min}$
- $C = \boldsymbol{u}'\boldsymbol{K}\boldsymbol{u}$
- $\boldsymbol{K}$ is an explicit function of $\boldsymbol{x}$: $\boldsymbol{K}(\boldsymbol{x}) = \sum_e \rho_e^p \boldsymbol{K}_e$
- $\boldsymbol{u}$ is an implicit function of $\boldsymbol{x}$: $\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{f}$
- Using product and chain rules: $\frac{dC}{dx_e} = -(1 - x_{min})p\rho_e^{p-1}\boldsymbol{u}'\boldsymbol{K}_e\boldsymbol{u}$

Implementation State

Modular experimental platform was setup:

- $>$ 5000 lines of Julia code across a few packages
  - TopOpt.jl (main, unpublished)
  - TopOptProblems.jl
  - LinearElasticity.jl
  - JuAFEM.jl
  - IterativeSolvers.jl
  - Preconditioners.jl
- Direct dependencies
  - **FEA**: JuAFEM.jl, Einsum.jl, IterativeSolvers.jl, Preconditioners.jl, StaticArrays.jl
  - **Optimization**: Optim.jl, MMA.jl
  - **Visualization**: Plots.jl, Makie.jl

Supported Features

- Finite element modelling
  1. **Material**: linearly elastic.
  2. **Mesh**: homogeneous 2D or 3D unstructured mesh of tri, quad, tetra or hexa elements.
  3. **Boundary conditions**: nodal and face Neumann and Dirichlet boundary conditions.
  4. **Import**: model can be imported from .inp files.
  5. **Analysis**: compliance, stress and buckling analysis.

Supported Features

- Linear system solver
  1. Direct sparse solver
  2. Assembly-based CG method
  3. Matrix-free CG method
- Eigenvalue solver
  1. Assembly-based LOBPCG method

Supported Features

- Topology optimization
  1. Chequerboard filter for unstructured meshes
  2. Can fix some cells as black or white
  3. Compliance objective
  4. Volume constraint
  5. SIMP
     - MMA.jl [3]
     - Continuation SIMP
  6. Soft-kill BESO [1]
  7. Genetic Evolutionary Structural Optimization (GESO) [2]