Topology Optimization and JuMP

Immense Potential and Challenges

Mohamed Tarek Mohamed

School of Engineering and Information Technology
UNSW Canberra

June 28, 2018
Introduction

Mohamed Tarek Mohamed

Topology Optimization and JuMP
About Me

- First year PhD student at UNSW Canberra
  - Multidisciplinary design optimization lab
  - Supervisor: Tapabrata Ray

- Background:
  - BSc mechanical engineering
  - MSc industrial engineering

- Research interests:
  1. Topology optimization algorithms
  2. Topology optimization and finite element modelling
  3. Multigrid methods and scalable topology optimization

- NumFOCUS GSoC student
  - Locally optimal block preconditioned conjugate gradient (LOBPCG) in IterativeSolvers.jl
  - Buckling analysis
Structure Design

F = 1 N

60mm

20mm
Structure Design

![Diagram of a design structure with labels for rows and columns. The diagram shows a complex pattern with varying thicknesses and connections between different sections.](image-url)
Problems
Problem Families

Variables:

1. Each optional mesh element corresponds to a variable,
2. Can be binary or continuous (variable thickness sheet)

Objectives:

1. Compliance minimization,
2. Material volume/cost minimization,
3. Maximum stress minimization, or
Problem Families

Constraints:

1. Volume constraint,
2. Maximum compliance constraint,
3. Maximum displacement constraint,
4. Local/global stress constraints,
5. Fatigue constraints,
6. Global stability constraints, and/or
7. Others.

![Diagram of beam with forces and dimensions]

$F = 1 \text{ N}$
Problem Families

**Mechanical systems:**

1. Linear, elastic, quasi-static system,
2. Nonlinear, compliant mechanism,
3. Nonlinear, elasto-plastic system,
4. Linear/nonlinear vibrating system, or
5. Others.

![Mechanical System Diagram](image-url)
**Problem Families**

**Loads:**

1. Single or multiple,
2. Static or dynamic,
3. Deterministic or stochastic, and
4. Design-dependent or design-independent.

![Diagram of a rectangular structure with loads](image)
Pipelines
## Topology Optimization Pipelines

**User’s Pipeline**

1. **Problem context definition**
   - Initial design in mesh form
   - Boundary conditions
   - Fixed cells, not allowed to change
   - Defined programmatically or through .inp and similar files

2. **Objective and constraint selection**

3. **Topology optimization algorithm**
   - Nested Analysis and Design (NAND), or
   - Simultaneous Analysis and Design (SAND)
Topography Optimization Pipelines

- Nested Analysis and Design (NAND) Pipeline
  1. Decide material distribution variables
     - Does this element exist or not?
     - Binary ∈ \{0, 1\}, or relaxed ∈ [0, 1]
  2. FEA
     - Cannot fully remove an element (numerical instability)
     - \(x_{soft} = x(1 - x_{min}) + x_{min}, \ \ x_{min} = 0.001\)
     - Makes use of matrix-free linear system and eigenvalue solvers
     - Can be GPU-accelerated or distributed on many computers
  3. Objective and constraint values and derivatives
     - Adjoint method: differentiating through the analysis equations
     - When binary constraints are relaxed, "penalized" variables, i.e. \(x_{penal} = x_{soft}^p\) are often used, for some known penalty typically \(1 < p \leq 5\).
  4. Update material distribution and repeat
     - Optimization magic!
Simultaneous Analysis and Design (SAND) Pipeline

1. FEA
   - Formulate the analysis equations as constraints
   - Analysis variables are decision variables

2. Optimization modelling
   - Write the composite analysis-design problem as a bigger optimization problem
   - Material distribution decision variables $x$
   - Analysis decision variables, e.g. nodal displacements $u$
   - Analysis constraints, e.g. $Ku = f$ and $K = \sum_{e} x_{\text{penal},e} K_e$
   - Design constraints, e.g. $\sigma_e^V \leq \sigma_y, \forall e$

3. Optimization magic!
   - Single pass
Examples
Example Problem 1

- Volume constrained compliance minimization
  - Analysis: 1b and 1c
  - Design: 1a and 1d
  - Chek filter not shown

\[
\text{minimize} \quad C = u^T Ku \quad (1a)
\]
\[
\text{subject to} \quad Ku = f, \quad (1b)
\]
\[
K = \sum_{e} \rho_e^p K_e, \quad (1c)
\]
\[
\sum_{e} \nu_e x_e \leq V, \quad (1d)
\]
\[
x_e \in \{0, 1\} \quad \forall e \quad (1e)
\]

\(C\): Compliance, convex in \(x\)

\(u\): Displacements

\(K\): Global stiffness matrix

\(f\): Load vector

\(K_e\): Element stiffness matrix \(e\)

\(\nu_e\): Volume of element \(e\)

\(x_e\): Does element \(e\) exist?

\(\rho_e\): Soft \(x_e := x_e(1 - x_{\text{min}}) + x_{\text{min}}\)

\(V\): Volume threshold

\(p\): Known as the "penalty"

typically \(\in [1, 5]\)
Example Problem 2

- Stress constrained volume minimization
  - $\sigma_{ij}^e$ is the stress tensor inside element $e$ \textbf{linear} in $u$
  - $(\sigma_v^e)^2 x_e \leq \sigma_y^2 x_e$ is also a valid constraint since $\sigma_v^e \geq 0$ and $x_e$ is binary
  - $(\sigma_v^e)^2 x_e \leq \sigma_y^2 x_e$ is \textbf{bi-convex} in $u$ and $x_e$

\[
\text{minimize} \quad \sum_e v_e x_e \\
\text{subject to} \\
(1b), \\
(1c), \\
\sigma_v^e x_e \leq \sigma_y x_e \quad \forall e, \\
x_e \in \{0, 1\} \quad \forall e
\]

\[
\sigma_v^e := \left( \frac{1}{2}(\sigma_{11}^e - \sigma_{22}^e)^2 + \frac{1}{2}(\sigma_{22}^e - \sigma_{33}^e)^2 + \frac{1}{2}(\sigma_{33}^e - \sigma_{11}^e)^2 + 3(\sigma_{12}^e)^2 + 3(\sigma_{23}^e)^2 + 3(\sigma_{31}^e)^2 \right)^{\frac{1}{2}} \quad \forall e
\]

$\sigma_y$: yield stress of the material
Example Problem 3

- Buckling constrained volume minimization
  - Positive semidefinite constraint
  - \( K \) is a **linear** function of \( x \)
  - \( K_\sigma \) is a **bi-linear** function of \( u \) and \( x \)

\[
\begin{align*}
\text{minimize} \quad & \sum_e v_e x_e \\
\text{subject to} \quad & (1b), \\
& (1c), \\
& K_\sigma = \sum_e x_e \int_{\Omega_e} G_e^T \psi_e G_e dV, \\
& K + \lambda_s K_\sigma \succeq 0, \\
& x_e \in \{0, 1\} \quad \forall e
\end{align*}
\]

- \( K_\sigma \): Stress stiffness matrix
- \( \lambda_s \): Load multiplier under which design must be stable
- \( \sigma^e \): matrix form of \( \sigma_{ij}^e \) from the previous slide
- \( \psi_e := \text{kron}(I_{\dim \times \dim}, \sigma^e) \)
- \( G^e \): basis function derivatives of element \( e \) arranged in a special order
Algorithms
Algorithm Classification Tree

Acronyms:

- **SAND**: Simultaneous analysis and design
- **NAND**: Nested analysis and design
- **MINL-SDP**: Mixed integer nonlinear and semidefinite programming
- **INL-SDP**: Integer nonlinear and semidefinite programming (no continuous variables)
- **NL-SDP**: Nonlinear and semidefinite programming
- **SIMP**: Solid isotropic material with penalization
- **BESO**: Bi-directional evolutionary structural optimization [1]
Algorithm Classification Tree

Single/Multi Objective

Exact

SAND

MINL-SDP

Binary

Incomplete

MINL-SDP

Heuristic

NAND

INL-SDP

Continuous

BESO

Incomplete

INL-SDP

SAND

Binary

Others

SIMP

Continuous

Others

NAND

Incomplete

NL-SDP

Others

NL-SDP

Mohamed Tarek Mohamed

Topology Optimization and JuMP
Why am I here?
Next Generation Topology Optimization

- Continuous and binary variables
- Flexible constraint handling
  - Block constraints with Jacobian of fixed sparsity pattern
  - Bi-linear, bi-convex and nonlinear constraints
  - Conic constraints
  - Partial structure, e.g. some bi-linear and some bi-convex
- Linear time and memory complexity
  - Can have 100s of millions of variables
- Numerically robust to scaling
- Scalable optimization pipeline (pre-processor and solver)
  - Efficiently GPU-accelerated
  - Efficiently distributed to multiple machines
- Single- and multi- objective
Possible with Julia’s optimization ecosystem?
Next Generation Topology Optimization

What can I offer?

- Prayers!
- Oh and I am ready to code (after paper submissions and GSoC!).
Demo
Further Readings I


Questions?
Method of moving asymptotes (MMA)

- Most popular nonlinear programming algorithm used in topology optimization
- Sequential convex programming
- First order approximation of $f$ with respect to $\frac{1}{x-L}$ or $\frac{1}{U-x}$, whichever is convex given the sign of $f'(x)$
- Originally proposed in [3]
- Later improved and similar algorithms were proposed in [4]
- Dual algorithm is fully separable so it can be GPU-accelerated and distributed
- Only handles inequality constraints
Adjoint method

- $\rho_e = x_e(1-x_{\text{min}}) + x_{\text{min}}$
- $C = u'Ku$
- $K$ is an explicit function of $x$: $K(x) = \sum_e \rho^p_e K_e$
- $u$ is an implicit function of $x$: $K(x)u(x) = f$
- Using product and chain rules: $\frac{dC}{dx_e} = -(1-x_{\text{min}})p\rho^{-1}_e u'K_e u$
Modular experimental platform was setup:

- > 5000 lines of Julia code across a few packages
  - `TopOpt.jl` (main, unpublished)
  - `TopOptProblems.jl`
  - `LinearElasticity.jl`
  - `JuAFEM.jl`
  - `IterativeSolvers.jl`
  - `Preconditioners.jl`

- Direct dependencies
  - **FEA**: `JuAFEM.jl`, `Einsum.jl`, `IterativeSolvers.jl`, `Preconditioners.jl`, `StaticArrays.jl`
  - **Optimization**: `Optim.jl`, `MMA.jl`
  - **Visualization**: `Plots.jl`, `Makie.jl`
Finite element modelling

1. **Material**: linearly elastic.
2. **Mesh**: homogeneous 2D or 3D unstructured mesh of tri, quad, tetra or hexa elements.
3. **Boundary conditions**: nodal and face Neumann and Dirichlet boundary conditions.
4. **Import**: model can be imported from .inp files.
5. **Analysis**: compliance, stress and buckling analysis.
Supported Features

- Linear system solver
  1. Direct sparse solver
  2. Assembly-based CG method
  3. Matrix-free CG method

- Eigenvalue solver
  1. Assembly-based LOBPCG method
Supported Features

- Topology optimization
  1. Chequerboard filter for unstructured meshes
  2. Can fix some cells as black or white
  3. Compliance objective
  4. Volume constraint
  5. SIMP
     - MMA.jl [3]
     - Continuation SIMP
  7. Genetic Evolutionary Structural Optimization (GESO) [2]