ProxSDP.jl: New developments on Semidefinite Programming in Julia/JuMP

Mario Souto and Joaquim Dias Garcia

March 19, 2019
Unique games conjecture

- Unique Games Conjecture: For a large class of problems, even finding an approximate solution is NP-hard.

- If the UGC is true, for a large class of problems, no polynomial-time algorithm can be better than $\frac{1}{2}$.
First Big Steps Toward Proving the Unique Games Conjecture

The latest in a new series of proofs brings theoretical computer scientists within striking distance of one of the great conjectures of their discipline.
Unique games conjecture

**Computational Complexity**

First Big Steps Toward Proving the Unique Games Conjecture

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*Do you think the unique games conjecture is true or false?* (en.m.wikipedia.org)

Submitted 1 year ago by 744196884
3 comments share save hide give award report crosspost

All 3 comments
Sorted by: best

*[-] Rioghasarig* 6 points 1 year ago
Yes
permalink embed save give award

*[-] GNULinuxProgrammer* 1 point 1 year ago
It seems to me that the same philosophical reasons as to why P is probably != NP can be applied to this conjecture.
permalink embed save give award

*[-] 744196884* [3] 2 points 1 year ago
But if that were true then the academic community wouldn't be evenly divided on whether the unique games conjecture were true or false. Compare it to p vs np where most people think p != np
Applications

- Control problems;

- Robust structural design (e.g. truss topology);

- Eigenvalue optimization problems;

- Relaxations for combinatorial problems (e.g. Max-Cut, graph coloring, traveling salesman, Max-Sat, ...);

- Optimal power flow relaxation;

- Machine Learning (matrix completion, robust PCA, kernel learning).
ALGORITHMS

A Classical Math Problem Gets Pulled Into the Modern World

A century ago, the great mathematician David Hilbert posed a probing question in pure mathematics. A recent advance in optimization theory is bringing Hilbert’s work into a world of self-driving cars.

MATHEMATICS

A New Tool to Help Mathematicians Pack

Improvements in how densely spheres and other shapes can be packed together could lead to advances in materials science, deep space communication and theoretical physics.

Quanta magazine
Why isn’t SDP widely used?

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  - Changing with the adoption of chordal decomposition;
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- Formulating the problem as a SDP may not always be straightforward:
  - Solved by modern modeling frameworks (JuMP.jl and others);

- State-of-the-art solvers are yet unable to solve large SDP problems.
Any SDP with $m$ constraints admits a solution with rank at most $\sqrt{2m}$ (Barvinok-Pataki 1995/98);
Motivation - Low-rank structure

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- In practice, several SDP problems admits even lower rank solutions;

- Interior points methods frequently compute the full rank solution;

- Low-rank structure is usually exploited as a matrix factorization (Burer-Monteiro 2003):

\[
X = V^T V \quad \text{where} \quad V \in \mathbb{R}^{k \times n} \quad \text{and} \quad k \text{ is the target rank.}
\]
Recap from JuMPdev 2018...

https://github.com/mariohsouto/ProxSDP.jl
Semidefinite Programming

- Primal:

  \[
  \begin{align*}
  \text{minimize} & \quad \operatorname{tr}(CX) \\
  \text{subject to} & \quad M(X) = b, \\
  & \quad X \succeq 0.
  \end{align*}
  \]

  where

  \[
  M(X) = \begin{bmatrix}
  \operatorname{tr}(M_1X) \\
  \operatorname{tr}(M_2X) \\
  \vdots \\
  \operatorname{tr}(M_mX)
  \end{bmatrix}.
  \]

- Problem data: \( M_1, \ldots, M_m, C \in \mathbb{S}^n, b \in \mathbb{R}^m \) and \( h \in \mathbb{R}^p \).
Optimality condition

\[ 0 \in \partial \text{tr}(CX) + \partial I_{\mathbb{R}_+^n}(X) + M^T(\partial I_{=b}^\leq (M(X))). \]

- Introducing an auxiliary variable \( y \in \mathbb{R}^{p+m} \):

\[ 0 \in \partial \text{tr}(CX) + \partial I_{\mathbb{R}_+^n}(X) + M^T(y), \]
\[ y \in \partial I_{=b}^\leq (M(X)). \]

- By definition, \( y \) is the dual variable associated with the linear constraints;

- If strong duality holds, any \((X^*, y^*)\) satisfying the inclusion above is the optimal primal-dual pair.
Algorithm PD-SDP

while $\epsilon_k^{\text{comb}} > \epsilon_{\text{tol}}$ do

\[
X^{k+1} \leftarrow \text{proj}_{S^n_+}(X^k - \tau(M^T(y^k) + C))
\]  \quad \triangleright \text{Primal step}

\[
y^{k+1/2} \leftarrow y^k + \sigma M((1 + \theta)X^{k+1} - \theta X^k)
\]  \quad \triangleright \text{Dual step part 1}

\[
y^{k+1} \leftarrow y^{k+1/2} - \sigma \text{proj}_{b}(y^{k+1/2}/\sigma)
\]  \quad \triangleright \text{Dual step part 2}

end while

return $(X^{k+1}, y^{k+1})$
Computational bottleneck

The computational complexity of each iteration of PD-SDP is $\mathcal{O}(n^3)$;
Computational bottleneck

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- The spectral decomposition can be prohibitive even for medium scale problems;
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The spectral decomposition can be prohibitive even for medium scale problems;

Can be reduced to $O(n^2r)$, if one knows the target rank $r$ \textit{a priori} to each iteration.
Low-rank approximation

▶ Truncated projection onto the positive semidefinite cone:

\[
aproj_{S^n_+}(X, r) = \sum_{i=1}^{r} \max\{0, \lambda_i\} u_i u_i^T,
\]

▶ From (Eckart–Young–Mirsky theorem 1936), the approximation error can be bounded as

\[
\left\| \proj_{S^n_+}(X) - \aproj_{S^n_+}(X, r) \right\|^2_F \leq (n - r) \max\{\lambda_r, 0\}.
\]
LR-PD-SDP

Algorithm LR-PD-SDP

while $(n - r) \lambda_r > \epsilon_{\lambda}$ do

while $\epsilon_{\text{comb}}^k > \epsilon_{\text{tol}}$ and $\epsilon_{\text{comb}}^k < \epsilon_{\text{comb}}^{k-\ell}$ do

$X^{k+1} \leftarrow \text{aproj}_{S_+^n}(X^k - \tau(M^T(y^k) + C), r)$ \hspace{1cm} ▶ Approx. primal step

$y^{k+1/2} \leftarrow y^k + \sigma M((1 + \theta)X^{k+1} - \theta X^k)$ \hspace{1cm} ▶ Dual step part 1

$y^{k+1} \leftarrow y^{k+1/2} - \sigma \text{proj}_b(y^{k+1/2}/\sigma)$ \hspace{1cm} ▶ Dual step part 2

end while

$r \leftarrow 2r$ \hspace{1cm} ▶ Target-rank update

end while

return $(X^{k+1}, y^{k+1})$
Street-fighting optimization

▶ Algorithmic

- Use adaptive step size for primal and dual update. Use heuristic for balance residuals;

- **Linesearch** for selecting over-relaxation parameter as large as possible.

▶ Computational

- Arpack eig function might fail. Limit the number of iterations, choose tolerance accordingly;

- Can use MKL if available.
Adding other cones and inequalities

Algorithm LR-PD-SDP

\[ \text{while } (n - r) \lambda_r > \epsilon \lambda \text{ do} \]

\[ \text{while } \epsilon^k_{\text{comb}} > \epsilon_{\text{tol}} \text{ and } \epsilon^k_{\text{comb}} < \epsilon^k_{\text{comb}} \text{ do} \]

\[ X^{k+1} \leftarrow \text{aproj}_K(X^k - \tau(M^T(y^k) + C), r) \] \quad \triangleright \text{Approx. primal step} \\
\[ y^{k+1/2} \leftarrow y^k + \sigma M((1 + \theta)X^{k+1} - \theta X^k) \] \quad \triangleright \text{Dual step part 1} \\
\[ y^{k+1} \leftarrow y^{k+1/2} - \sigma \text{proj}_{b \leq h}(y^{k+1/2}/\sigma) \] \quad \triangleright \text{Dual step part 2} \\

end while

\[ r \leftarrow 2r \] \quad \triangleright \text{Target-rank update} \\

end while

\[ \text{return } (X^{k+1}, y^{k+1}) \]
Graph equipartition problem

<table>
<thead>
<tr>
<th>n</th>
<th>sdplib</th>
<th>SCS</th>
<th>CSDP</th>
<th>MOSEK</th>
<th>PD-SDP</th>
<th>LR-PD-SDP</th>
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<td>2.1</td>
<td>0.9</td>
<td>3.4</td>
<td>0.9</td>
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<tr>
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<td>6.1</td>
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<td>113.5</td>
<td>22.5</td>
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</table>

Table: Comparison of running times (seconds) for the SDPLIB’s graph equipartition problem instances.
Table: Comparison of running times (seconds) for randomized network localization problem instances.

<table>
<thead>
<tr>
<th>n</th>
<th>SCS</th>
<th>CSDP</th>
<th>MOSEK</th>
<th>PD-SDP</th>
<th>LR-PD-SDP</th>
</tr>
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<tbody>
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<td>0.2</td>
<td>0.1</td>
<td>0.5</td>
<td>0.6</td>
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<tr>
<td>100</td>
<td>0.8</td>
<td>4.5</td>
<td>0.9</td>
<td>6.1</td>
<td>1.6</td>
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<td>150</td>
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<td>32.3</td>
<td><strong>6.1</strong></td>
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<td>300</td>
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<td>timeout</td>
<td>85.2</td>
<td>96.6</td>
<td><strong>13.5</strong></td>
</tr>
</tbody>
</table>

Sensor network localization
MIMO experiments

<table>
<thead>
<tr>
<th>n</th>
<th>SCS</th>
<th>CSDP*</th>
<th>MOSEK</th>
<th>PD-SDP</th>
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<tbody>
<tr>
<td>100</td>
<td>1.5</td>
<td>1.2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>500</td>
<td>277.8</td>
<td>27.4</td>
<td>2.3</td>
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<td>1.1</td>
</tr>
<tr>
<td>1000</td>
<td>timeout</td>
<td>97.2</td>
<td>15.6</td>
<td>16.5</td>
<td>4.7</td>
</tr>
<tr>
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<td>timeout</td>
<td>473.6</td>
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<td>timeout</td>
<td>timeout</td>
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<td>122.1</td>
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<td>timeout</td>
<td>timeout</td>
<td>976.8</td>
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<tr>
<td>5000</td>
<td>timeout</td>
<td>timeout</td>
<td>timeout</td>
<td>timeout</td>
<td>472.4</td>
</tr>
</tbody>
</table>

Table: Running times (seconds) for MIMO detection with high SNR.
Conclusion

▶ Achievements:

- Primal-dual method for solving SDP;

▶ Future ideas:

- Explore properties of intermediate low-rank feasible solution;
- Combine proposed method with chordal sparsity techniques;
- Exploit low rank structure of other problems (SOS, AC relaxation...)

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